# Spatiotemporal chaos in disordered nonlinear lattices

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# All started back in 2008

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# Outline

- **Brief overview of the dynamics of 1D Disordered lattices:**
  - ✓ **The quartic disordered Klein-Gordon (DKG) model**
  - ✓ **The disordered discrete nonlinear Schrödinger equation (DDNLS)**
  - ✓ **Different dynamical regimes**
- **Symplectic Integrators – Tangent Map Method**
- **Numerical investigation of chaos**
  - ✓ **Maximum Lyapunov Exponent (MLE): strength of chaos**
  - ✓ **Deviation Vector Distributions (DVDs): mechanisms of chaotic spreading**
  - ✓ **Frequency Map Analysis (FMA): characteristics of spatiotemporal evolution of chaos**
  - ✓ **Generalized Alignment Index (GALI): localized vs. spreading chaos**
- **Chaotic behavior of the DKG and DDNLS models in 2 spatial dimensions**

# The 1D disordered Klein – Gordon model (1D DKG)

$$H_{1K} = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2$$

with **fixed boundary conditions** $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: **W** and the **total energy E.** $\tilde{\varepsilon}_l$ **chosen uniformly from** $\left[\frac{1}{2},\frac{3}{2}\right]$.

**Linear case** (neglecting the term $u_l^4/4$)

**Ansatz:** $u_l=A_l \exp(i\omega t)$. **Normal modes (NMs)** $A_{\nu,l}$ **- Eigenvalue problem:**

$$\lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \text{ with } \lambda = W\omega^2 - W - 2, \quad \varepsilon_l = W(\tilde{\varepsilon}_l - 1)$$

# The 1D disordered discrete nonlinear Schrödinger equation (1D DDNLS)

We also consider the system:

$$H_{1D} = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right)$$

where $\varepsilon_l$ **chosen uniformly from** $\left[-\frac{W}{2},\frac{W}{2}\right]$ and $\beta$ **is the nonlinear parameter.**

**Conserved quantities:** The energy and the norm $S = \sum_l |\psi_l|^2$ of the wave packet.

# Distribution characterization (1D case)

We consider normalized **energy distributions** $\xi_l \equiv \frac{E_l}{\sum_m E_m}$

with $E_l = \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{4W}\left(u_{l+1} - u_l\right)^2$ for the DKG model,

and **norm distributions** $\xi_l \equiv \frac{|\psi_l|^2}{\sum_m |\psi_m|^2}$ for the DDNLS system.

**Second moment:** $m_2 = \sum_{l=1}^{N} \left(l - \bar{l}\right)^2 \xi_l$ with $\bar{l} = \sum_{l=1}^{N} l \xi_l$

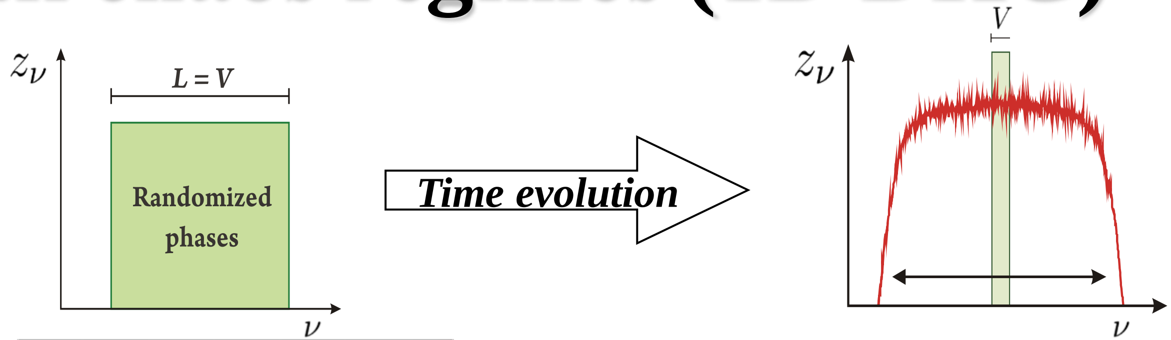
**Participation number:** $P = \frac{1}{\sum_{l=1}^{N} \xi_l^2}$

measures the number of stronger excited sites in $\xi_l$.

Single site $P=1$. Equipartition of energy $P=N$.

# **Strong** and **weak** chaos regimes (1D DKG)

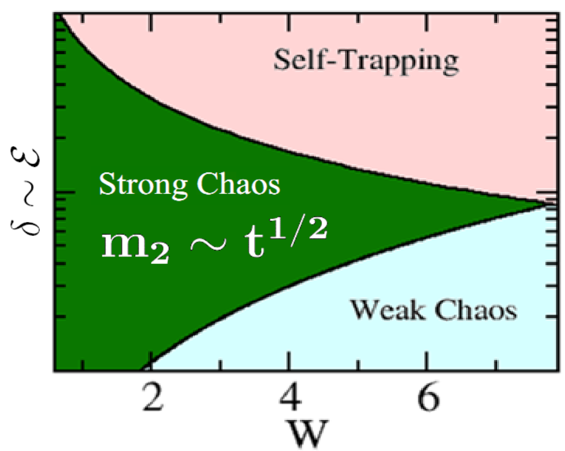
**We consider compact initial wave packets of width *L* [Flach et al. PRL (2009) –S. et al PRE (2009) – Laptyeva et al., EPL (2010) – Bodyfelt et al., PRE (2011)]**

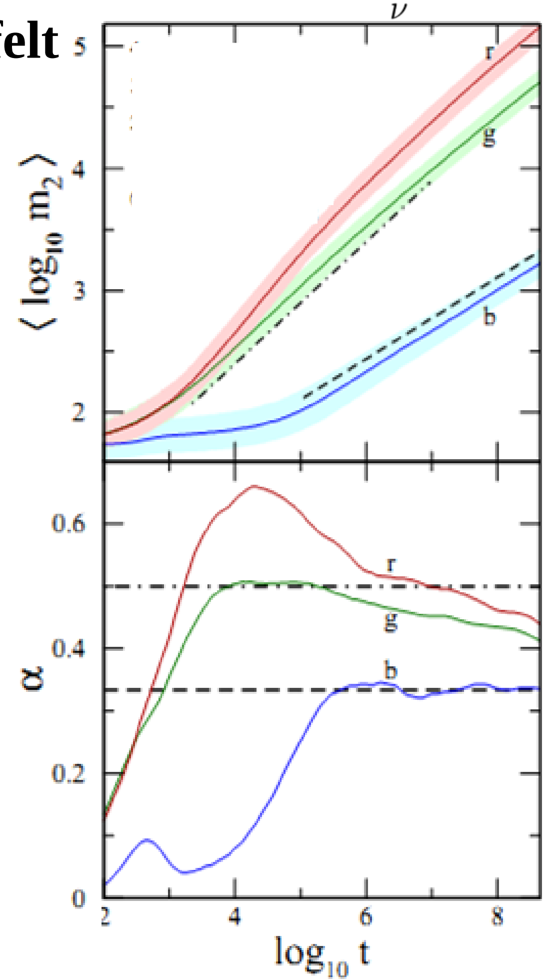



**H= 0.01, 0.2, 0.75**

**W=4**

**Average over 1000 realizations!**




**α=1/2**

**α=1/3**

$$\alpha(\log t) = \frac{d\langle \log m_2 \rangle}{d \log t}$$

# Maximum Lyapunov Exponent (MLE)

**Chaos: sensitive dependence on initial conditions.**

**Roughly speaking, the MLE of a given orbit characterizes the mean exponential rate of divergence of trajectories surrounding it.**

**Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector (small perturbation) from it v(0).**
**Then the mean exponential rate of divergence is:**

$$\text{MLE} = \lambda_1 = \lim_{t\to\infty} \Lambda(t) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

$\lambda_1 = 0 \rightarrow$ **Regular motion** $(\Lambda \propto t^{-1})$

$\lambda_1 > 0 \rightarrow$ **Chaotic motion**

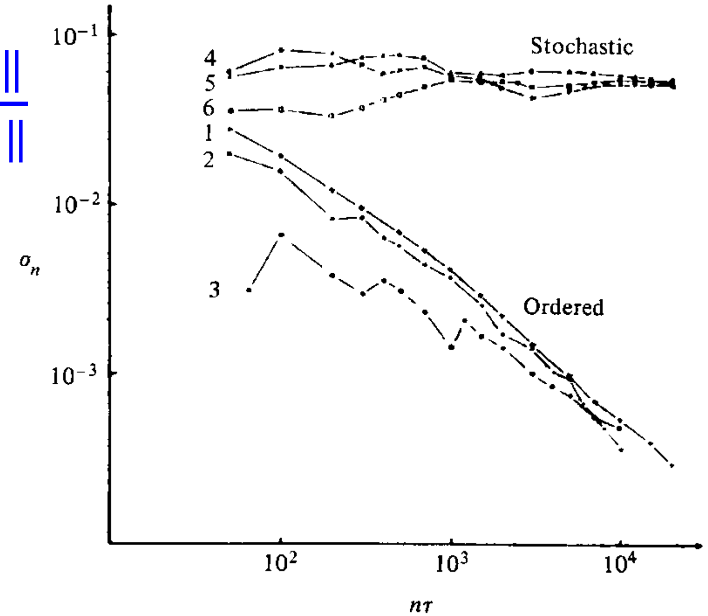



Figure 5.7. Behavior of $\sigma_n$ at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

# Symplectic integration

**We apply the 2-part splitting integrator ABA864 [Blanes et al., Appl. Num. Math. (2013) – Senyange & S., EPJ ST (2018)] to the DKG model:**

$$H_{1K} = \sum_{l=1}^{N} \left( \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left( u_{l+1} - u_l \right)^2 \right)$$

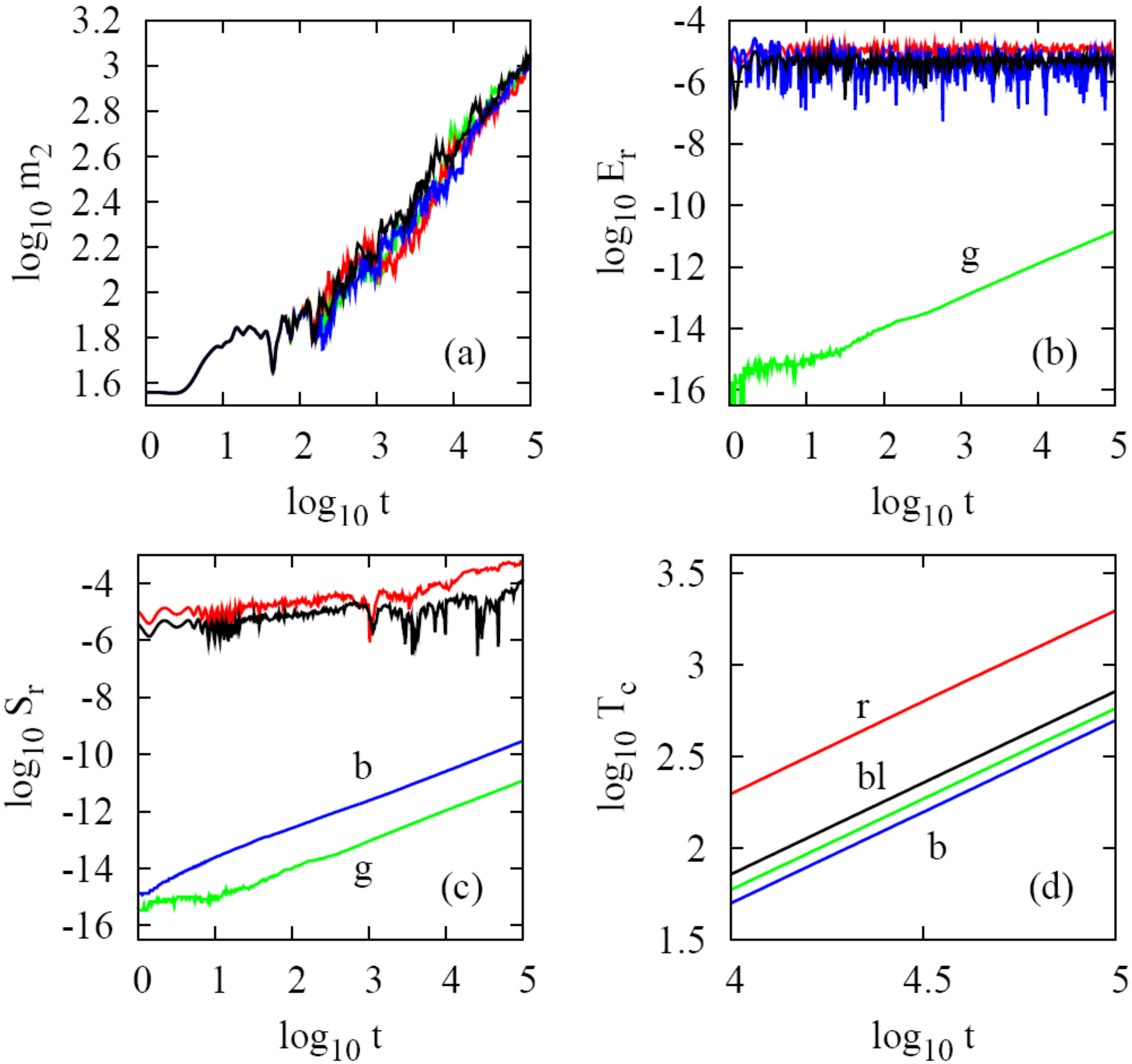
**and the 3-part splitting integrator $ABC^6_{[SS]}$ [S. et al., Phys. Let. A (2014) – Gerlach et al., EPJ ST (2016) – Danieli et al., MinE (2019) ] to the DDNLS system:**

$$H_{1D} = \sum_l \varepsilon_l \left|\psi_l\right|^2 + \frac{\beta}{2} \left|\psi_l\right|^4 - \left( \psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l \right), \qquad \psi_l = \frac{1}{\sqrt{2}} \left( u_l + i p_l \right)$$

$$H_{1D} = \sum_l \left( \frac{\varepsilon_l}{2} \left( u_l^2 + p_l^2 \right) + \frac{\beta}{8} \left( u_l^2 + p_l^2 \right)^2 - u_n u_{n+1} - p_n p_{n+1} \right)$$

**By using the so-called Tangent Map method we extend these symplectic integration schemes in order to integrate simultaneously the variational equations [S. & Gerlach, PRE (2010) – Gerlach & S., Discr. Cont. Dyn. Sys. (2011) – Gerlach et al., IJBC (2012)].**

# $2^{nd}$ order integrators: Numerical results (1D DDNLS)




ABC$^2$ $\tau$=0.005

SS$^2$ $\tau$=0.02

DOP853 $\delta=10^{-16}$

SIFT$^2$ $\tau$=0.05

**$E_r$: relative energy error**

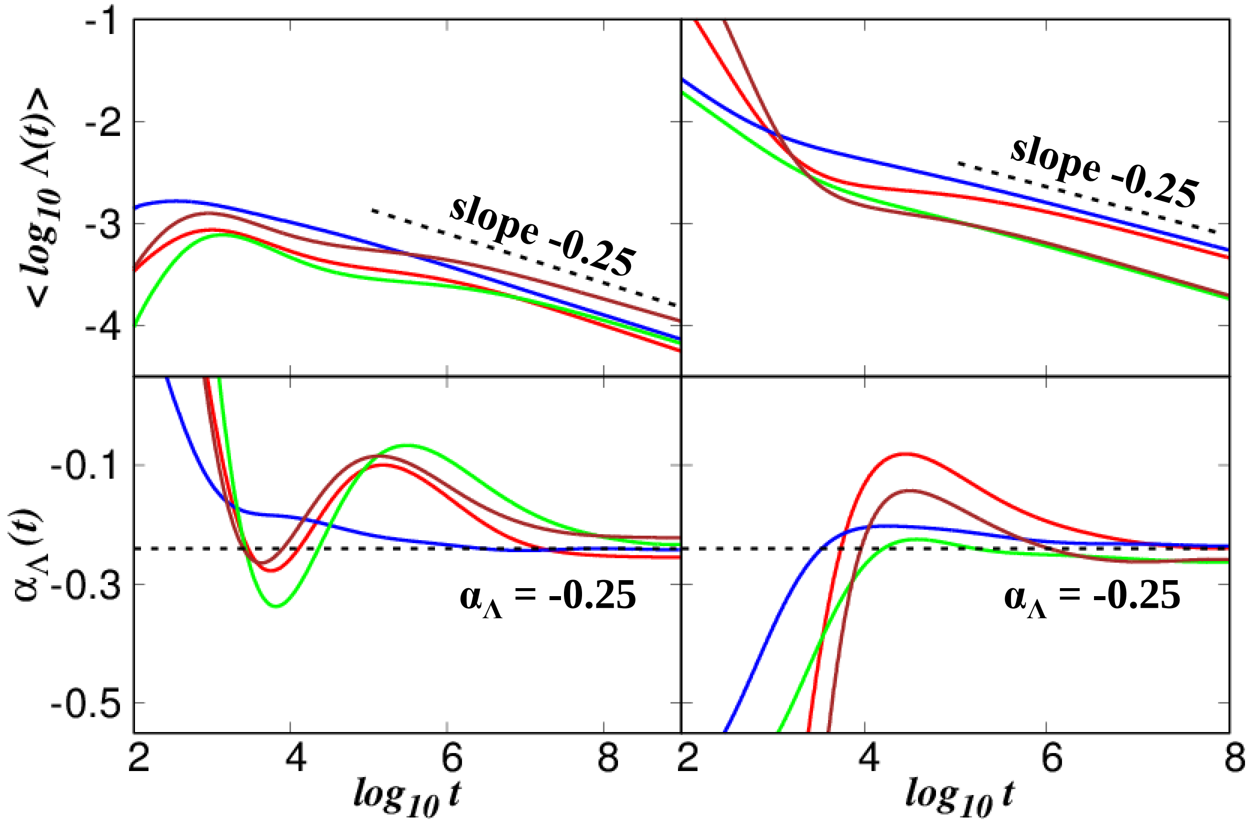
**$S_r$: relative norm error**

**$T_c$: CPU time (sec)**

**S. et al., Phys. Lett. A (2014)**

# Weak Chaos: 1D DKG and 1D DDNLS

1D DKG




1D DDNLS

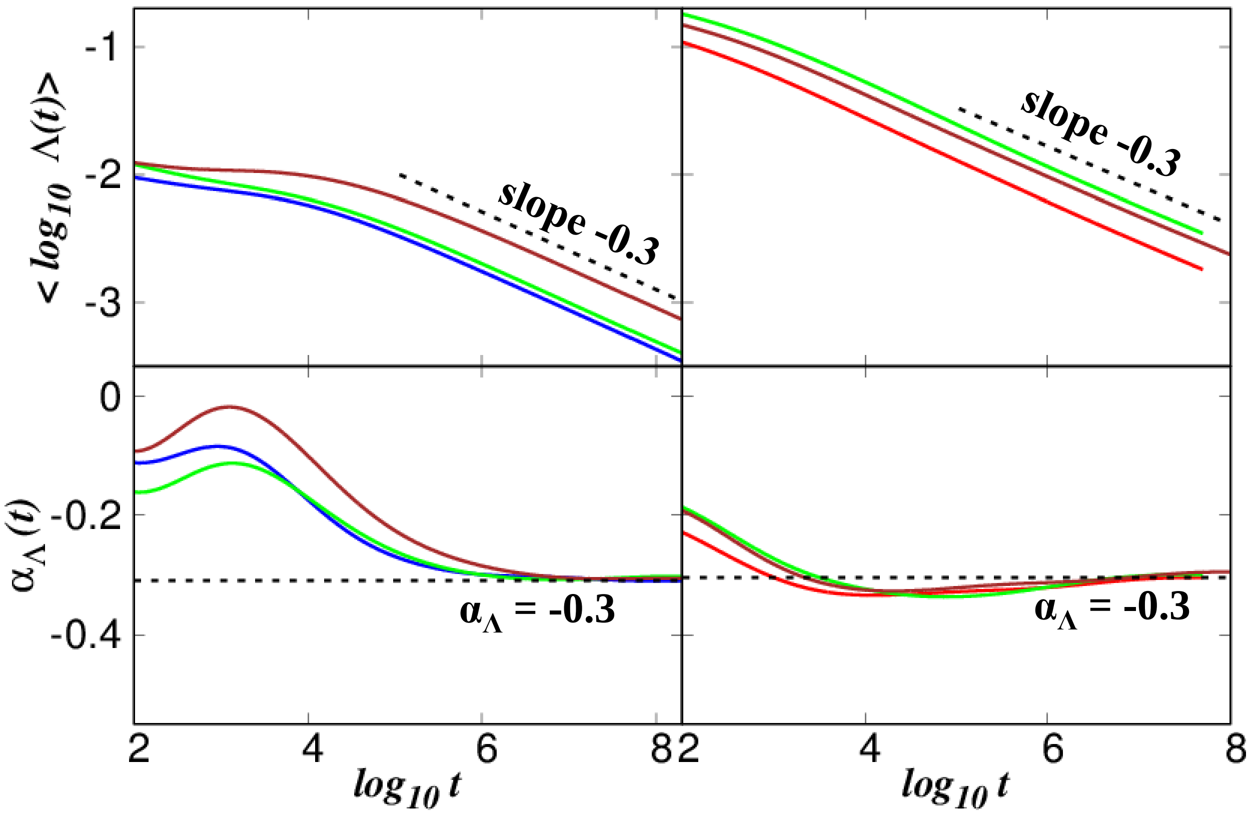
Average over 100 realizations [Senyange, Many Manda & S., PRE (2018)]

| 1D DKG | 1D DDNLS |
| --- | --- |
| Block excitation (L=37 sites) $H_{1K}$=0.37, W=3 | Block excitation (L=21 sites) β=0.04, W=4 |
| Single site excitation $H_{1K}$=0.4, W=4 | Single site excitation β=1, W=4 |
| Block excitation (L=21 sites) $H_{1K}$=0.21, W=4 | Single site excitation β=0.6, W=3 |
| Block excitation (L=13 sites) $H_{1K}$=0.26, W=5 | Block excitation (L=21 sites) β=0.03, W=3 |

1D DKG model also studied in S. et al., PRL (2013)

# Strong Chaos: 1D DKG and 1D DDNLS

1D DKG




1D DDNLS

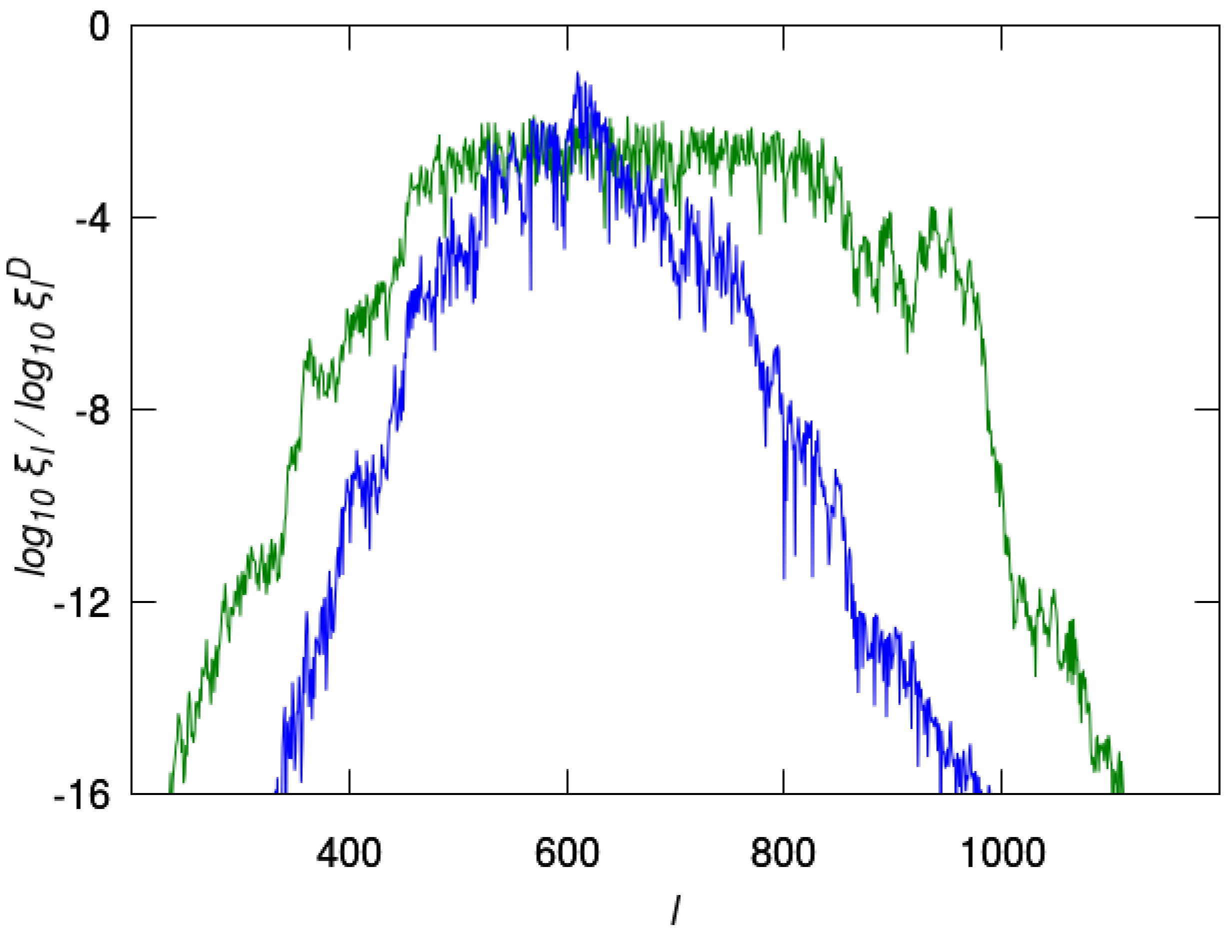
**Average over 100 realizations [Senyange, Many Manda & S., PRE (2018)]**

**Block excitation (L=83 sites) $H_{1K}$=0.83, W=2** **Block excitation (L=21 sites) β=0.62, W=3.5**

**Block excitation (L=37 sites) $H_{1K}$=0.37, W=3** **Block excitation (L=21 sites) β=0.5, W=3**

**Block excitation (L=83 sites) $H_{1K}$=0.83, W=3** **Block excitation (L=21 sites) β=0.72, W=3.5**

# Deviation Vector Distributions (DVDs)




**Energy**
**DVD**

**1D DKG**
**weak chaos**
**L=37 sites,**
**$H_{1K}$=0.37,**
**W=3**

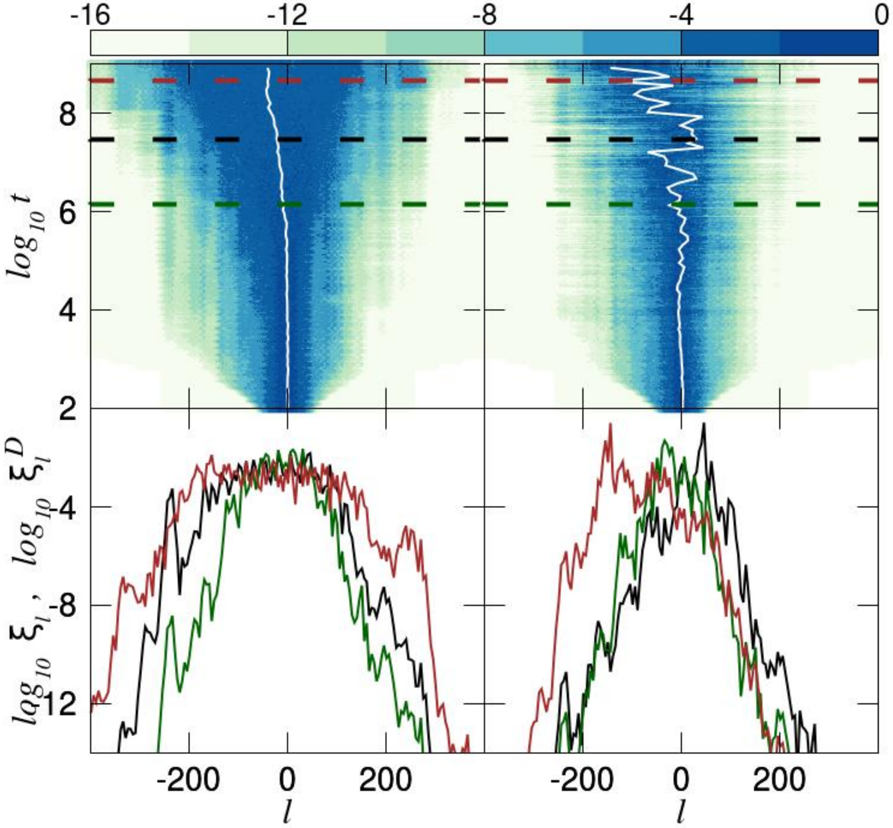
**Deviation vector:**
**$v(t)=(\delta u_1(t), \delta u_2(t),\ldots, \delta u_N(t), \delta p_1(t), \delta p_2(t),\ldots, \delta p_N(t))$**

**DVD:** $$\xi_l^D = \frac{\delta u_l^2 + \delta p_l^2}{\sum_l \left(\delta u_l^2 + \delta p_l^2\right)}$$

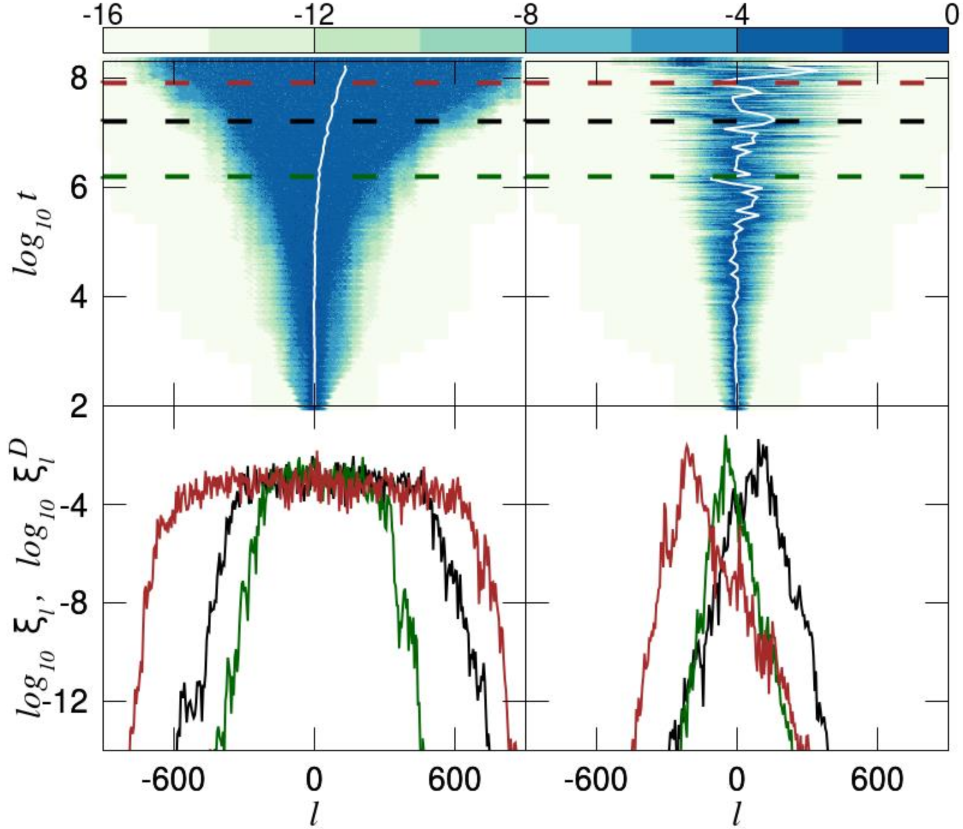
# DVDs: Weak and Strong Chaos

**Weak chaos (1D DKG)** | **Strong chaos (1D DKG)**

**Energy** **DVD**




**Energy** **DVD**




**W=3, L=37, H=0.37** | **W=3, L=83, H=8.3**

**Chaotic hot spots meander through the system, supporting the homogeneity of chaos inside the wave packet.**

# Frequency Map Analysis (FMA)

**Compute the fundamental frequencies, $f_1$ and $f_2$, of an observable related to the evolution of an orbit in two successive time windows of the same length, and check whether or not these frequencies change in time [Laskar, Icarus (1990) – Laskar et al., Physica D (1992) – Laskar, Physica D (1993) – Robutel & Laskar, Icarus (2000)].**

**Regular motion: The computed frequencies do not vary in time**

**Chaotic motion: The computed frequencies vary in time**

**For every lattice site *l* we compute the fundamental frequencies $f_{1l}$ and $f_{2l}$ for time windows of length $T = 6 \cdot 10^5$ time units and evaluate the relative change of these two frequencies:**

$$D_l = \left| \frac{f_{2l} - f_{1l}}{f_{1l}} \right|$$

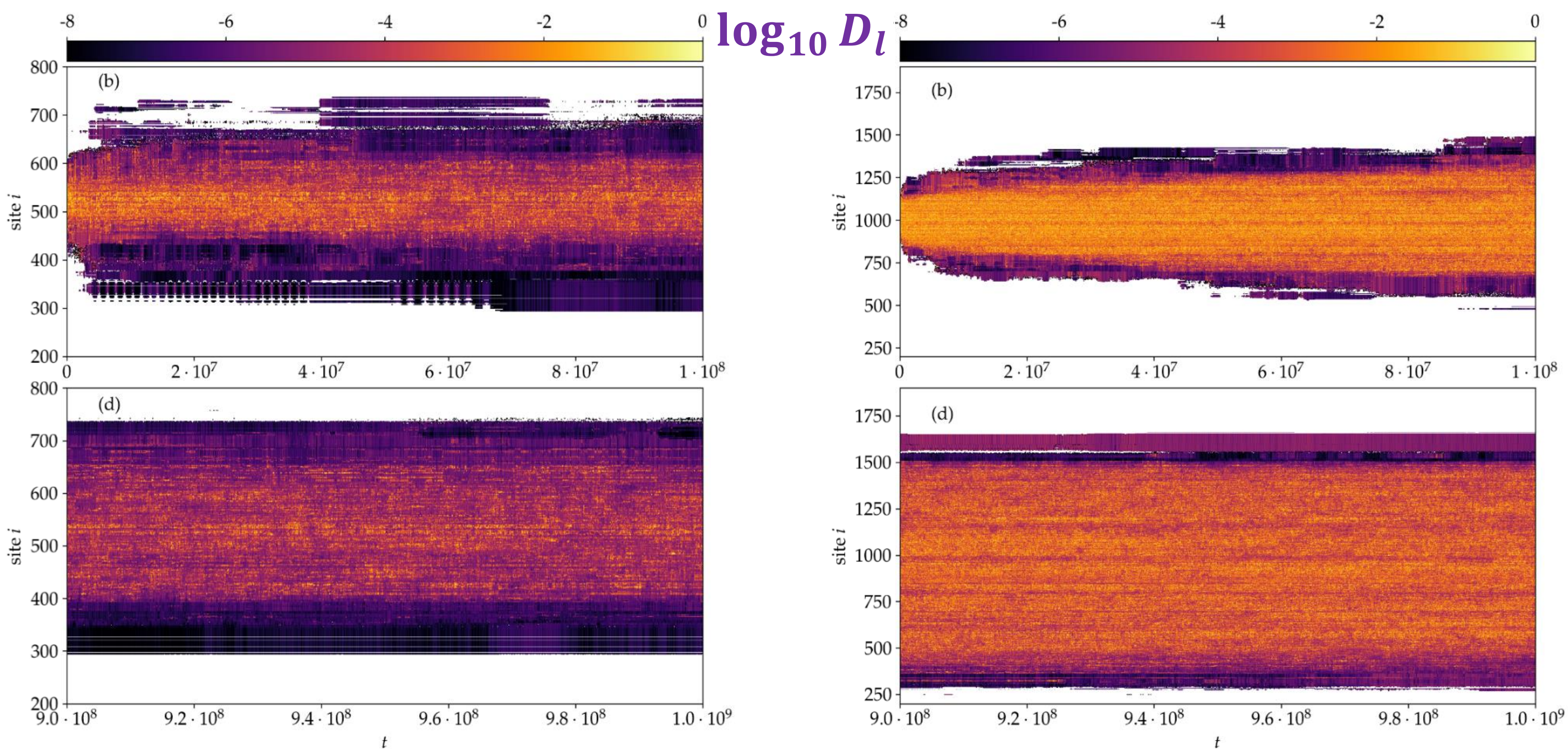
**Regular motion: small $D_l$ values**

**Chaotic motion: large $D_l$ values**

# FMA: Weak and Strong Chaos

**Weak chaos**
**L=1, H=0.4, W=4, N=999**

**Strong chaos**
**L=21, H=4.2, W=4, N=3499**

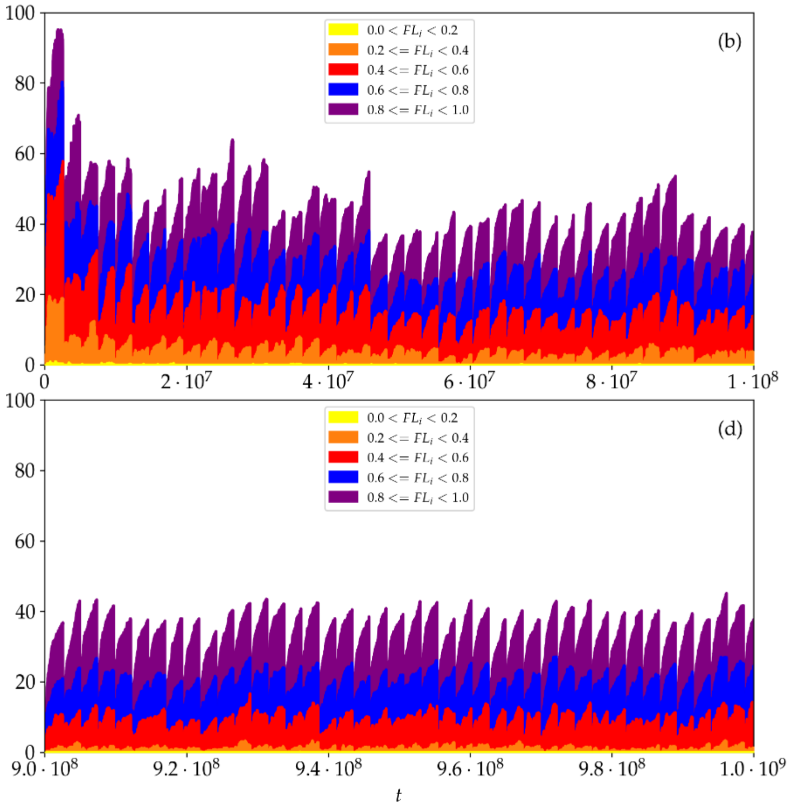



**Chaotic behavior appears at the central regions of the wave packet, where the energy density is relatively large. The chaotic component of the wave packet is more extended in the strong chaos case [S. et al., IJBC (2022)]**

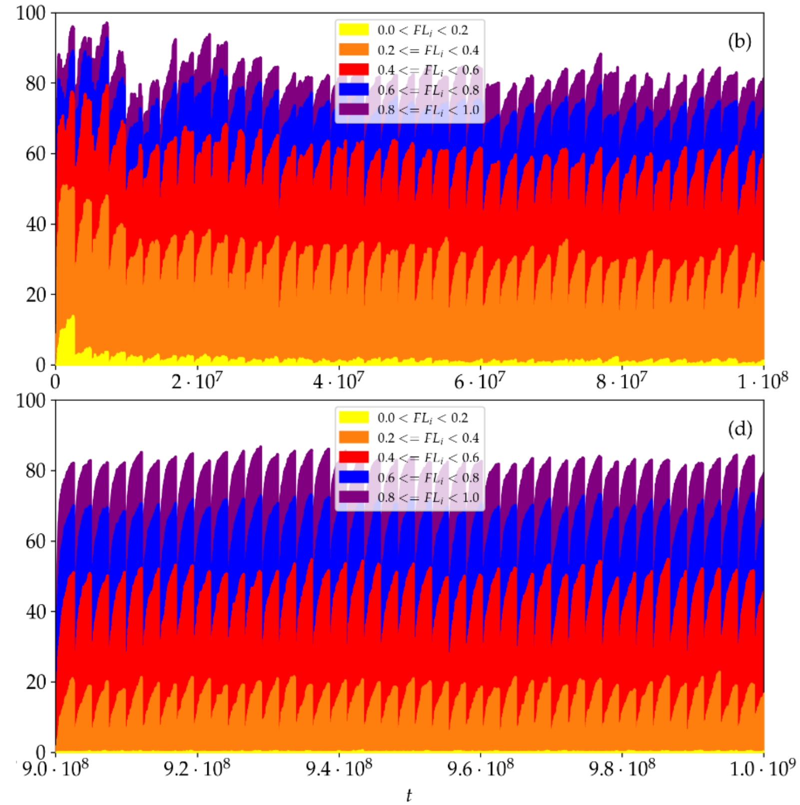
# Frequency Locking (FL)

**Accumulated percentages $P_{FL}$ of sites with values in a particular $FL$ range**

**Weak chaos**




**Strong chaos**




**The fraction of sites behaving chaotically is much larger in the strong chaos regime.**

**The percentage of strongly chaotic sites (having $FL_l < 0.4$) is about 5 times larger for strong chaos.**

**For both spreading regimes, the fraction of highly chaotic oscillators ($FL_l < 0.4$) decreases in time, although the percentage of chaotic sites remains practically constant.**

# The Generalized Alignment Index (GALI)

**In the case of an N degree of freedom Hamiltonian system we follow the evolution of k deviation vectors with 2≤k≤2N, and define [S. et al., Physica D, (2007)] the Generalized Alignment Index (GALI) of order k :**

$$GALI_k(t) = \|\hat{v}_1(t) \wedge \hat{v}_2(t) \wedge \ldots \wedge \hat{v}_k(t)\|$$

**where** $\hat{v}_1(t) = \frac{v_1(t)}{\|v_1(t)\|}$.

**Chaotic motion:** $GALI_k(t) \propto e^{-[(\lambda_1-\lambda_2)+(\lambda_1-\lambda_3)+\ldots+(\lambda_1-\lambda_k)]t}$

**with $\lambda_1$, $\lambda_2$, …, $\lambda_k$ being the first k largest Lyapunov exponents.**

**Regular motion:** **When the motion occurs on an N-dimensional torus [S. et al., Eur. Phys. J. Sp. Top. (2008)]:**

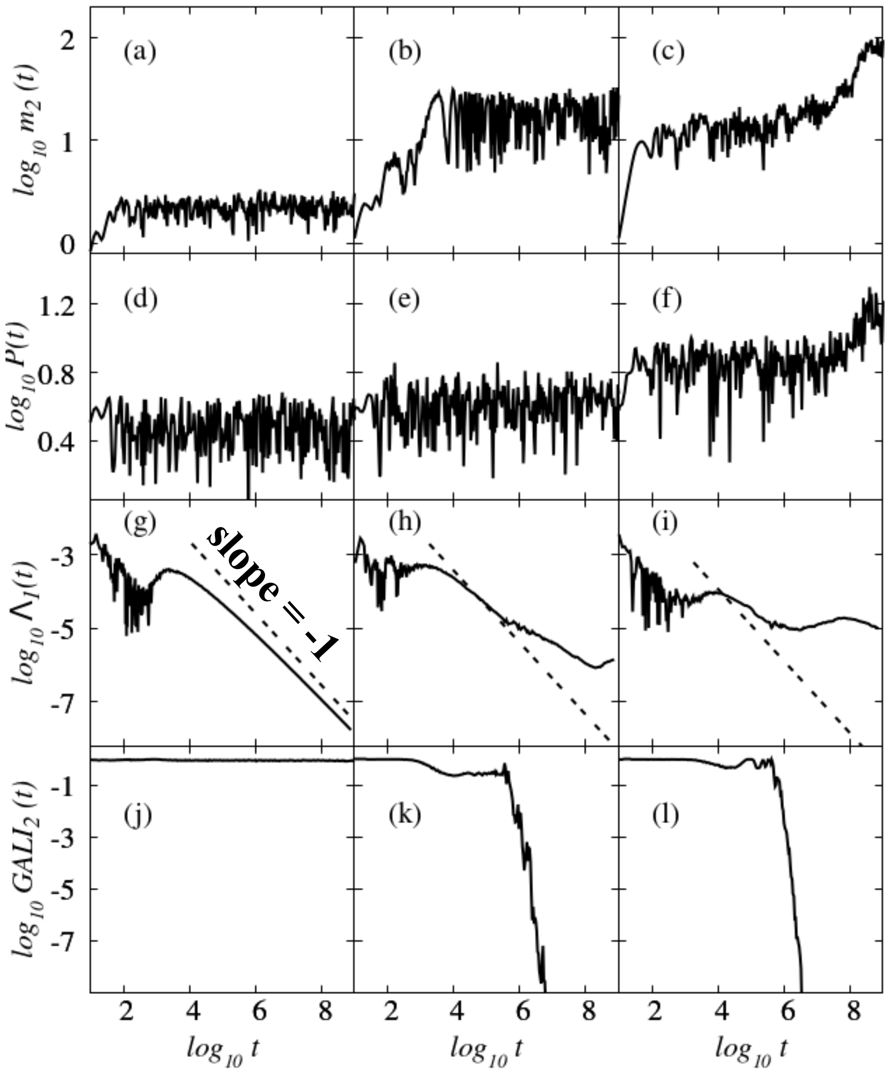
$$GALI_k(t) \propto \text{constant, if } 2 \leq k \leq N$$

**Here we consider GALI$_2$ (k=2) which is equivalent to the Smaller Alignment Index (SALI) [S, J. Phys A (2001)].**

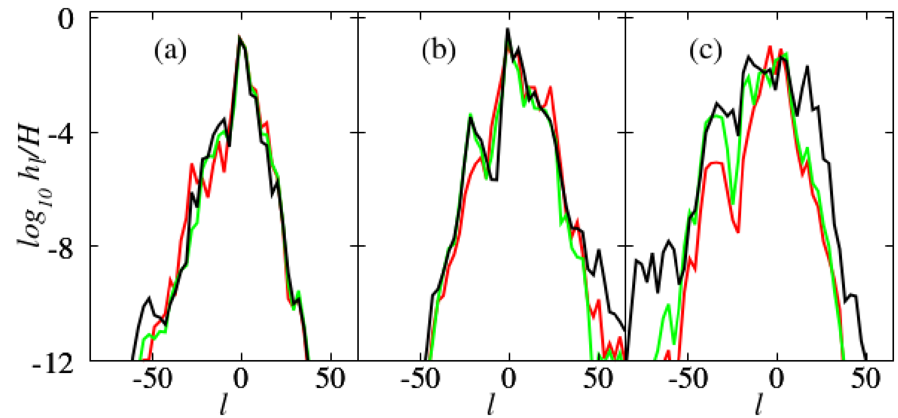
# Regular vs. chaotic (localized or spreading) motion

## Different disorder realizations can exhibit different behaviors.

**Regular motion** | **Localized chaos** | **Spreading chaos**




**Regular motion** | **Localized chaos** | **Spreading chaos**



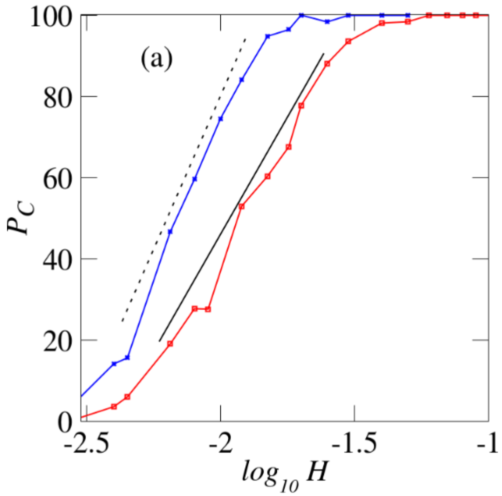

$t = 10^5, 10^7, 10^9$

**Single site excitations, L=1, for W=6, H=0.02 [Senyange & S., Physica D (2022)].**

**The $GALI_2$ can identify chaos much more clearly than the MLE.**

# Decreasing nonlinearity

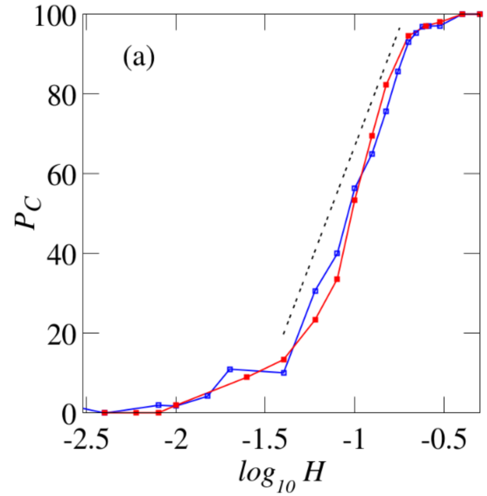
**Single site excitations**



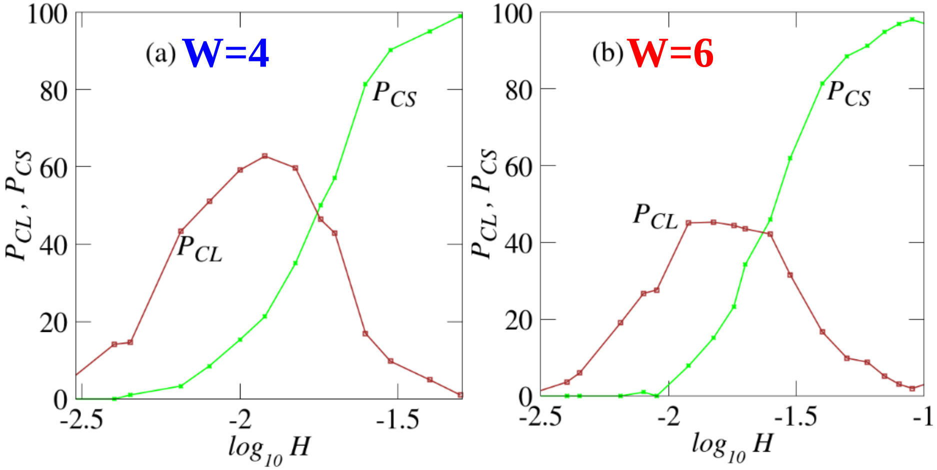

**$P_C$: % of chaotic orbits**

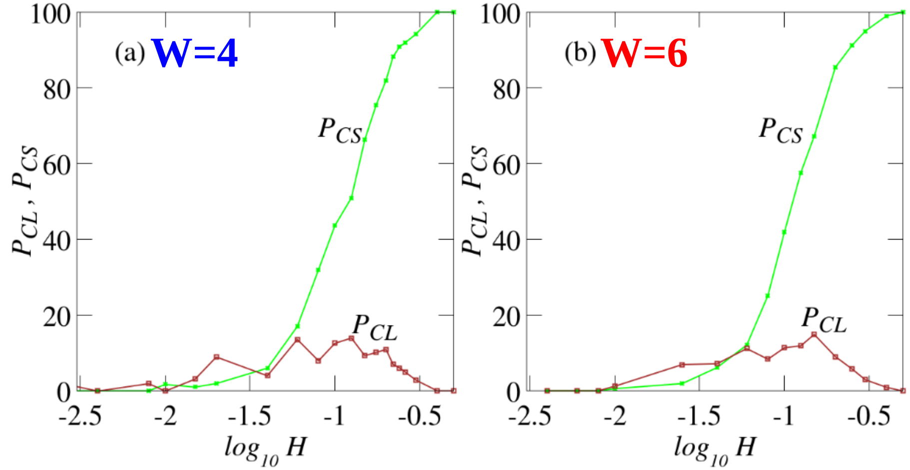
**W=4, W=6**

**Single mode excitations**




**$P_{CL}$: % of localized chaos $P_{CS}$: % of spreading chaos**




**Energy thresholds for transition to regular motion and to spreading chaos are lower for single site excitations which permit mode interactions [Senyange & S., Physica D (2022)].**

# The 2D DKG model

$$H_{2K}=\sum_{l,m}\left\{\frac{p_{l,m}^2}{2}+\frac{\tilde{\varepsilon}_{l,m}}{2}u_{l,m}^2+\frac{u_{l,m}^4}{4}+\right.$$
$$\left.+\frac{1}{2W}\left[\left(u_{l,m+1}-u_{l,m}\right)^2+\left(u_{l+1,m}-u_{l,m}\right)^2\right]\right\}$$

**Again we have fixed boundary conditions and $\tilde{\varepsilon}_{l,m}$ are chosen uniformly in $\left[\frac{1}{2},\frac{3}{2}\right]$.**

# The 2D DDNLS system

$$H_{2D}=\sum_{l,m}\left\{\frac{\varepsilon_{l,m}}{2}\left(u_{l,m}^2+p_{l,m}^2\right)+\frac{\beta}{8}\left(u_{l,m}^2+p_{l,m}^2\right)^2-\right.$$
$$\left.-\left(u_{l,m+1}u_{l,m}+u_{l+1,m}u_{l,m}+p_{l,m+1}p_{l,m}+p_{l+1,m}p_{l,m}\right)\right\}$$

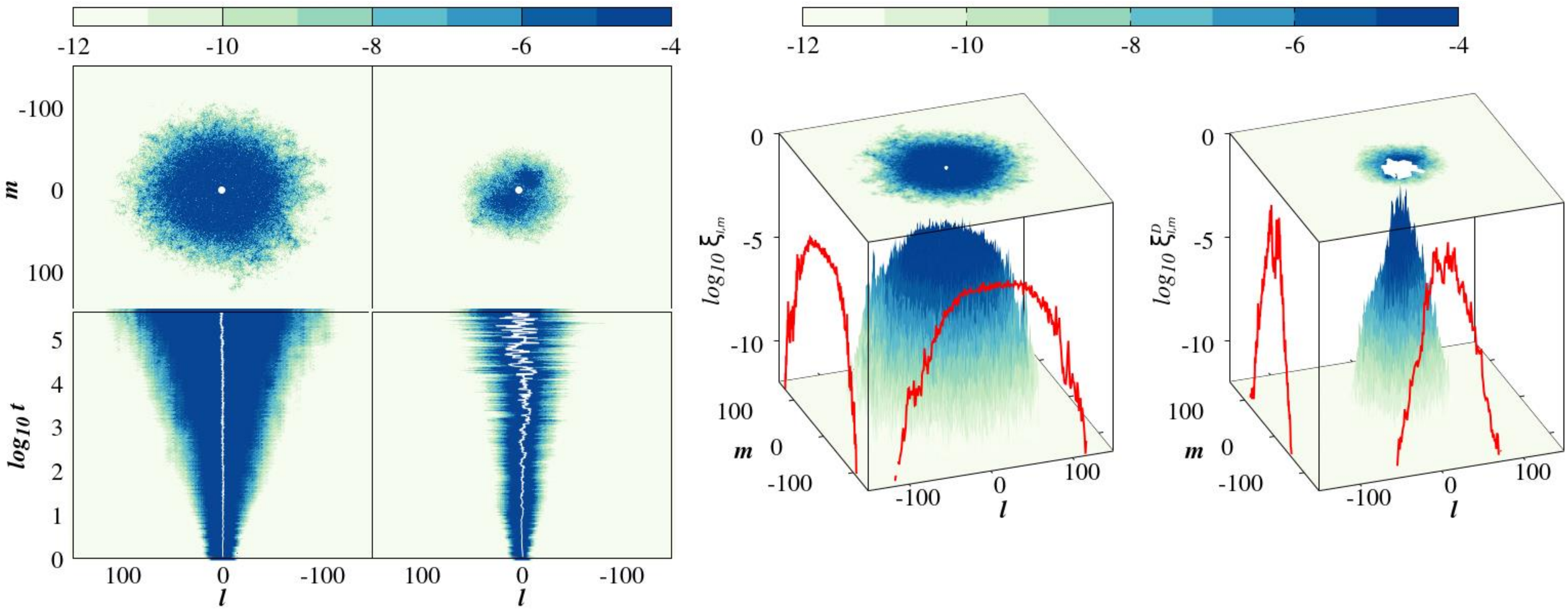
**Again $\varepsilon_l$ are chosen uniformly from $\left[-\frac{W}{2},\frac{W}{2}\right]$ and $\beta$ is the nonlinear parameter.**

**Conserved quantities: The energy $H_{2D}$ and the norm $S=\sum_{l,m}\frac{u_{l,m}^2+p_{l,m}^2}{2}$**

# 2D: Deviation Vector Distributions (DVDs)

**2D DDNLS: strong chaos**

**L=15, W=12, $\beta$=0.425, $s_{l,m}$=1, $H_{2D}$=1.32**




**Norm** **DVD** **Norm** **DVD**

# Dimension-independent scaling between chaoticity and spreading

Second moment: Theoretical predictions verified by numerical computations

$$m_2 \propto t^{a_m}$$

| $a_m$ | Weak | Strong |
|---|---|---|
| 1D | 1/3 | 1/2 |
| 2D | 1/5 | 1/3 |

| $a_\Lambda$ | Weak | Strong |
|---|---|---|
| 1D | -0.25 | -0.30 |
| 2D | -0.37 | -0.46 |

$$\Lambda \propto t^{a_\Lambda}$$

Finite time mLCE: Numerical computations

**For 1D and 2D systems there exists a uniform *scaling between the wave packet's spreading and its degree of chaoticity* indicating that nonlinear interactions of the same nature are responsible for the chaotic wave-packet spreading in both cases.**

$$\left.\frac{\Lambda(t)}{m_2(t)}\right|_{1D} = \left.\frac{\Lambda(t)}{m_2(t)}\right|_{2D}$$

**Weak chaos**

$$t^{-0.58} \approx t^{-0.57}$$

**Strong chaos**

$$t^{-0.80} \approx t^{-0.79}$$

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**Dynamics of disordered lattices**

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**Numerical integration of multidimensional Hamiltonian systems**

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**Chaos detection techniques**